NUMEROUS SERIES (III - PART)

ALTERNATIVE SERIES

Alternate series are series with variable signs of their members.

Forms are $a_1 - a_2 + a_3 - a_4 + ... = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$

DEF: (a) $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges **absolutely** if the series $\sum_{n=1}^{\infty}|a_n|$ converges

(b) $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges **conditionally** if it converges AND series $\sum_{n=1}^{\infty}|a_n|$ diverges

criteria:

Leibniz criteria:

Alternative series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges if $a_n > a_{n+1}$ for n=1,2,3... (Monotonically decreasing) and $\lim_{n\to\infty} a_n = 0$

Abelian criteria:

Series $\sum_{n=1}^{\infty} a_n b_n$ converges if :

i) $\sum_{n=1}^{\infty} a_n$ converges

ii) numbers b_n form monotonically limited series

Dirihle criteria:

Series $\sum_{n=1}^{\infty} a_n b_n$ converges if:

i) partial sums $S_n = \sum_{k=1}^{n} a_k$ are limited

ii) b_n monotonically approaches zero when $n \to \infty$

Theorem (often used in tasks)

If (a_n) is positive series such that $\boxed{\dfrac{a_n}{a_{n+1}} = 1 + \dfrac{p}{n} + o(\dfrac{1}{n^2})}$ when $n \to \infty$ then series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$:

i) converges if $p > 0$ and $\left\{ \begin{array}{l} \text{- converges **absolutely** if } p > 1 \\ \text{- converges **conditionally** if } 0 < p < 1 \end{array} \right\}$

ii) diverges if $p \leq 0$

Yet we should remember that:

- If series is absolutely convergent then it is convergent
- The sum of absolute convergent series does not depend on the order of addition of its members.
- The sum of the conditional convergent series by changing order of addition of its members may have an arbitrary value (Riemann's theorem)

EXAMPLES

Example 1.

Examine the convergence of series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

Solution:

Here is $a_n = \frac{1}{n}$

$n < n+1 \to \frac{1}{n} > \frac{1}{n+1}$ and we conclude that this is a monotonically decreasing series, and $\lim_{n \to \infty} \frac{1}{n} = \frac{1}{\infty} = 0$,

and the Leibniz criterion for this series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ tells us that he converges.

What about the absolute convergence?

Look $\sum_{n=1}^{\infty} |a_n|$. For our series it is $\sum_{n=1}^{\infty} \frac{1}{n}$, and we already said that it diverges, so series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is not absolutely convergent. **It is only conditionally convergent.**

Example 2.

Examine the convergence of series $\sum_{n=1}^{\infty}(-1)^n \frac{2}{\sqrt{n^2+2}+n}$

Solution:

First, we notice that $\frac{2}{\sqrt{n^2+2}+n} > 0$ for each n from set N

Further observe that:

$$n+1>n$$
$$(n+1)^2>n^2$$
$$(n+1)^2+2>n^2+2$$
$$\sqrt{(n+1)^2+2}>\sqrt{n^2+2}$$
$$\sqrt{(n+1)^2+2}+(n+1)>\sqrt{n^2+2}+n$$
$$\boxed{\frac{1}{\sqrt{(n+1)^2+2}+(n+1)}<\frac{1}{\sqrt{n^2+2}+n}} \rightarrow a_{n+1}<a_n$$

Therefore, it is a descending series, yet to find $\lim_{n\to\infty}\frac{1}{\sqrt{n^2+2}+n}=\frac{1}{\infty}=0$

So, series $\sum_{n=1}^{\infty}(-1)^n \frac{2}{\sqrt{n^2+2}+n}$ is convergent by Leibniz criteria.

To investigate the absolute convergence:

$$\sum_{n=1}^{\infty}\left|(-1)^n \frac{2}{\sqrt{n^2+2}+n}\right|=\sum_{n=1}^{\infty}\frac{2}{\sqrt{n^2+2}+n}$$

When $n\to\infty$ we think like this:

$$\frac{2}{\sqrt{n^2+2}+n}\sim\frac{2}{\sqrt{n^2}+n}\sim\frac{2}{n+n}\sim\frac{2}{2n}\sim\frac{1}{n}$$

So, this series is the same **"character"** as well as series $\sum_{n=1}^{\infty}\frac{1}{n}$, which is divergent.

We conclude that the initial series $\sum_{n=1}^{\infty}(-1)^n \frac{2}{\sqrt{n^2+2}+n}$ **conditionally** convergent, and $\sum_{n=1}^{\infty}\frac{2}{\sqrt{n^2+2}+n}$ diverges.

Example 3.

Examine the convergence of series $\sum_{n=1}^{\infty}(-1)^n \frac{n!}{n^n}$

Solution:

Come here immediately to investigate the absolute convergence $\sum_{n=1}^{\infty}\left|(-1)^n \frac{n!}{n^n}\right| = \sum_{n=1}^{\infty}\frac{n!}{n^n}$

We will use :

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}\frac{\cancel{(n+1)}\cdot\cancel{n!}}{\cancel{n!}}\cdot\frac{n^n}{\cancel{(n+1)}\,(n+1)^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n =$$

$$= \lim_{n\to\infty}\left(\frac{1}{\frac{n+1}{n}}\right)^n = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e}$$

Since this series converges absolutely, immediately conclude that the series $\sum_{n=1}^{\infty}(-1)^n \frac{n!}{n^n}$ converges.

Example 4.

Examine the convergence of series $\sum_{n=2}^{\infty}\frac{\cos\frac{\pi n^2}{n+1}}{\ln^2 n}$

Solution:

Here is our idea to use Abelian criteria:

Series $\sum_{n=1}^{\infty} a_n b_n$ converges if :

i) $\sum_{n=1}^{\infty} a_n$ converges

iii) numbers b_n form monotonically limited series

From trigonometry we know that:

$$\cos\frac{\pi n^2}{n+1} = (-1)^{n+1}\cos\frac{\pi}{n+1}$$

Now look at series $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}\cos\frac{\pi}{n+1}}{\ln^2 n}=\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\ln^2 n}\cdot\cos\frac{\pi}{n+1}$

Series $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{\ln^2 n}$ is convergent and $\cos\frac{\pi}{n+1}$ form a monotonic and limited series.

Example 5.

Examine the convergence of series $\sum_{n=1}^{\infty}\frac{\ln^{50} n}{n}\sin\frac{n\pi}{4}$

Solution:

Here we use Dirihle criteria:

Series $\sum_{n=1}^{\infty}a_n b_n$ converges if:

ii) partial sums $S_n=\sum_{k=1}^{n}a_k$ are limited

ii) b_n monotonically approaches zero when $n\to\infty$

We will use a result of the previous files: $\left|\sum_{k=1}^{n}\sin\frac{k\pi}{4}\right|<\frac{1}{\sin\frac{\pi}{8}}$

$b_n=\frac{\ln^{50} n}{n}$ monotonically approaches zero when $n\to\infty$

$$\lim_{n\to\infty}\frac{\ln^{50} n}{n}=\left(\frac{\infty}{\infty}\right)=50\lim_{n\to\infty}\frac{\ln^{49} n}{n}=50\cdot 49\lim_{n\to\infty}\frac{\ln^{48} n}{n}=etc.=0$$

So, series converges.

Example 6.

Examine the convergence of series $\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{(2n-1)!!}{(2n)!!}\right]^p$

Solution:

The idea is to do the examination of absolute convergence series $\sum_{n=1}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^p$

This task we have worked in one of the previous files:

$$\frac{a_n}{a_{n+1}}=\frac{\left[\frac{(2n-1)!!}{(2n)!!}\right]^p}{\left[\frac{(2n+1)!!}{(2n+2)!!}\right]^p}=\left[\frac{(2n-1)!!}{(2n+1)!!}\frac{(2n+2)!!}{(2n)!!}\right]^p=\left[\frac{(2n-1)!!}{(2n+1)(2n-1)!!}\frac{(2n+2)(2n)!!}{(2n)!!}\right]^p=\left[\frac{2n+2}{2n+1}\right]^p$$

Now pack a little the term and use binomial formula:

$$\left[\frac{2n+2}{2n+1}\right]^p=\left[\frac{2n+1+1}{2n+1}\right]^p=\left[1+\frac{1}{2n+1}\right]^p=$$

$$=\binom{p}{0}1^p(\frac{1}{2n+1})^0+\binom{p}{1}1^{p-1}(\frac{1}{2n+1})^1+\binom{p}{2}1^{p-2}(\frac{1}{2n+1})^2+...$$

$$=1+\frac{p}{2n+1}+\underline{\left|\frac{p(p+1)}{2(2n+1)^2}+o(\frac{1}{n^2})\right.}$$

$$=1+\frac{p}{2n+1}+o(\frac{1}{n^2})$$

$$=1+\frac{p}{2(n+\frac{1}{2})}+o(\frac{1}{n^2})$$

$$=1+\frac{p/2}{n+1/2}+o(\frac{1}{n^2})\quad \text{when } \ n\to\infty$$

$$=1+\frac{p/2}{n}+o(\frac{1}{n^2})$$

Now, we use:

If (a_n) is positive series such that $\boxed{\frac{a_n}{a_{n+1}}=1+\frac{p}{n}+o(\frac{1}{n^2})}$ when $n\to\infty$ then series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$:

iii) converges if $p>0$ and $\left\{\begin{array}{l}\text{- converges } \textbf{absolutely} \text{ if } p>1\\ \text{- converges } \textbf{conditionally} \text{ if } 0<p<1\end{array}\right\}$

iv) diverges if $p\le 0$

We have:

Series converges if $p/2>0\to p>0$ and $\left\{\begin{array}{l}\text{- converges absolutely for } p/2>1\to\boxed{p>2}\\ \text{- converges conditionally for } 0<p/2<1\to\boxed{0<p<2}\end{array}\right\}$

Series diverges for $p/2\le 0\to\boxed{p\le 0}$